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STABILITY OF HETEROGENEOUS SYSTEMS WITH COHERENT INTERPHASE BOUNDARIES*

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The necessary conditions for stability (the sufficient conditions for instability) are found for two-phase configurations with coherent transformation boundaries. The phases (assumed isotropic) of the same chemical substance differ by elastic moduli of different orders. In addition, the "natural" deformation (ND) of the transition between the phase reference configurations is assumed to be small.

By utilizing the asymptotically small ND of the phase transformation it is possible to find the physical parameters and equations of the interphase surfaces for the equilibrium configurations as series expansions in a small parameter in the neighbourhood of the reference states. In the case of coherent transitions, the homogeneous stress-deformed state of one phase is uniquely determined by the given homogeneous state of the adjacent phase, the orientation of the plane coherent boundary, and the ND of the transformation /1, 2/.

To analyse the stability conditions for the equilibrium of heterogeneous thermodynamic systems, an approach is used which is based on studying the non-negative definiteness of the second variation of the appropriate energy functional /3, 4/. The equilibrium stability of configurations in which the phases are in the homogeneous stress-deformed state and are separated by a plane interphase boundary is found by seeing the conditions for non-negative definiteness of the proper spectral value of a system of linear homogeneous partial differential equations with constant coefficients and with suitable boundary conditions.

Isothermal heterogeneous systems with isotropic incompressible phases will be considered here. In the context of an asymptotically small transformation ND, the solution of the spectral problem of /4/ is sought as series in a small parameter. In the lowest approximation in the two-dimensional case, for homogeneously deformed phases, this means that the problem can be greatly simplified and criteria can be found for the loss of stability of the plane coherent boundary. The stability question is uniquely solved when the stress-deformed state of one phase is known and the transformation ND is given. The equation is found for the critical deformations (the equation of the surface of neutral stability), which have the order of the transformation ND. The stabilizing and destablizing factors are found for the coherent boundaries. It is shown that, if, in the hydrostatically stressed state, there is a phase with a lower shift modulus in a two-phase equilibrium configuration with coherent boundary and ND of cubic expansion-compression, then the system is always unstable. These results are used to examine the local stability of some equilibrium configurations of heterogeneous systems with curved coherent boundaries between the inhomogeneously deformed phases.

The stability of a periodic structure consisting of alternating layers of two different phases is considered. The relevant dispersion equations and the equations of the surface of neutral stability are found. The necessary conditions for the stability of these structures to disturbances of the symmetric and antisymmetric type are stated. A special limiting case of this problem is the problem of the stability of the plane layer representing the embryo of a new phase in an unbounded elastic matrix.

The effect of external boundaries is exemplified by considering the stability of a new phase embryo on the surface of an elastic half-space (a rigid wall of free surface). In the rigid-wall case, long-wave disturbances do not lead to a loss of stability. In the free-surface case, instability under long-wave disturbances sets in if there is a non-zero jump of the principal stresses in a direction tangential to the interphase boundary.

1. The necessary conditions for the stability of systems with coherent interphase boundaries. Consider an isothermal heterogeneous system consisting of two elastic phases ("plus" and "minus") of the same chemical substance, which are separated by a coherent interphase boundary. There is no surface tension and there are no external fields of force.

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The reference conguration of the pluse phase is identified with the initial one-phase configuration /1/. We assume that, corresponding to transition to the reference configuration of the minus phase, we have a small affine ND, which causes displacements w_i of the material particles, which are given by the relation $w_i = \varepsilon\Delta_{ij}x^j$, $\Delta_{ij} \sim 1$, $\varepsilon \ll 1$, where x^j are the Lagrangian Cartesian coordinates in the initial configuration $(i, j = 1, 2, 3)$.

Assuming that the phases in the reference configurations are in the unstressed state at a temperature where the densities of the free energies of the phases per unit mass are the same, we can find the fields of the auxiliary particle displacements v^i and the equation of the interphase boundary x^i $(\xi^\alpha, \varepsilon)$ $(\xi^1, \xi^2$ are the surface coordinates) in the actual heterogeneous configuration, as series in the small parameter ε /1/.

$$v_\pm{}^i = \sum_{N=1}^{\infty} \varepsilon^N v^i_{N\pm}, \quad x^i(\xi^\alpha, \varepsilon) = \sum_{N=0}^{\infty} \varepsilon^N x^i_{\cdot N}(\xi^\alpha) \tag{1.1}$$

A spectral problem has been established /3, 4/ for studying the stability conditions for an equilibrium heterogeneous configuration in which the homogeneously deformed phases are separated by a plane interphase boundary. We denote by $\varkappa_{ij}$ the gradients, constant in each phase, of the displacements $v_{i,j}$ (a comma in front of a Latin subscript denotes differentiation with respect to the space coordinate of the initial configuration).

The solution of the spectral problem of /4/ is also sought in the form of the series

$$\pi = \sum_{N=0}^{\infty} \varepsilon^N \pi_N, \quad a^i(x^k) = \sum_{N=0}^{\infty} \varepsilon^N a_N{}^i, \quad c = \sum_{N=-1}^{\infty} \varepsilon^N c_N \tag{1.2}$$

Here, π is the eigenvalue of the spectral problem, a^i are the variations of the field of auxiliary displacements, and c is the "velocity" of the interphase surface of the pre-image in the initial configuration in the direction of the unit normal n^i, induced by a change of the variation parameter. In a non-trivial real field a^i, c, which belongs to an eigenvalue π and which satisfies the normalization condition (1.8) of /4/, the second variation of the free energy of the system takes extremal value, equal to π. Consequently, the eigenvalues π must be non-negative for the thermodynamic system to be stable.

In the lowest approximation in ε, the spectral problem of /4/ is

$$\bar\psi^{ijkl} a_{0k,\,lj} + \pi_0 a_0{}^i = 0 \tag{1.3}$$

$$[a_0{}^i] = -c_{-1}[v^i_{1\cdot,j} + \Delta_{\cdot j}{}^i]\, n^j$$

$$[\bar\psi^{ijkl} a_{0k,\,l}]\, n_j = c_{-1,\alpha} x^{\cdot\alpha}_{j\cdot 0} [\bar\psi^{ijkl} v_{1k,\,l}]$$

$$([\bar\psi^{ijkl} v_{1k,\,l} a_{0i,\,\alpha}] + [\bar\psi^{ijkl} v_{1k,\,l}(v_{1i,\,p} + \Delta_{ip})]\, n^p c_{-1,\,\alpha})\, x^{\cdot\alpha}_{j\cdot 0} =$$

$$[\bar\psi^{ijkl}(v_{1i,\,p} + \Delta_{ip})\, a_{0k,\,l}]\, n^p n_j$$

$$\bar\psi^{ijkl} \equiv \partial^2\bar\psi / \partial v_{i,\,j} \partial v_{k,\,l}$$

$$x^i_{\cdot\alpha} \equiv \partial x^i / \partial\xi^\alpha, \quad [a] \equiv a^+ - a^-$$

Here, ψ is the density of the free energy per unit mass, and the bar indicates that the derivative is evaluated for $\varepsilon = 0$; a comma in front of a Greek subscript indicates partial differentiation with respect to a surface coordinate. The first relation of (1.3) must hold in the relevant phases, and the rest on the interphase boundary.

For the isotropic phases, the components of the tensor $\bar\psi^{ijkl}$ are evaluated from /5/

$$m\bar\psi^{ijkl} = \lambda\delta^{ij}\delta^{kl} + \mu(\delta^{ik}\delta^{jl} + \delta^{il}\delta^{jk}) \tag{1.4}$$

Here, m is the mass density in the initial configuration, and $\lambda_\pm, \mu_\pm$ are the Lamé coefficients, evaluated in the reference configurations.

We shall confine ourselves to the case of plane deformation, taking the fields Δ_{i3}, v_3, a_3 equal to zero (instead of a_{0i}, v_{1i}, and π_0 we henceforth use the notation a^i, v_i, and π), and the fields v_1, v_2, a_1, a_2 as independent of x^3. We shall also assume that the phases are isotropic and incompressible, i.e., the coefficient λ tends to infinity, while $\varkappa_{\cdot i}{}^i$ and $a^i_{\cdot,i}$ vanish ($i = 1, 2$; $i = 1$ refers to the coordinate x, and $i = 2$, to z). We denote the finite quantity $\lambda_\pm \varkappa^i_{\cdot i\pm}$ by $p_\pm$, and $\lambda_\pm a^i_{\cdot,i\pm}$ by $p_\pm'(x, z)$. Augmenting system (1.3) by the incompressibility condition, then eliminating c_{-1} and using the equilibrium conditions (see e.g., the first of (2.10) of /1/), the system becomes

a) inside the phases

$$c^2_{\perp\pm}\left(\frac{\partial^2 a_1{}^\pm}{\partial x^2} + \frac{\partial^2 a_1{}^\pm}{\partial z^2}\right) - m^{-1}\frac{\partial p_\pm'}{\partial x} + \pi a_1{}^\pm = 0 \tag{1.5}$$

$$c^2_{\perp\pm}\left(\frac{\partial^2 a_2{}^\pm}{\partial x^2} + \frac{\partial^2 a_2{}^\pm}{\partial z^2}\right) - m^{-1}\frac{\partial p_\pm'}{\partial z} + \pi a_2{}^\pm = 0$$

$$\frac{\partial a_1^{\pm}}{\partial x}+\frac{\partial a_2^{\pm}}{\partial z}=0, \quad c_{\perp\pm}\equiv(\mu_{\pm}m^{-1})^{1/2}$$

b) on the surface (pre-image of the interphase boundary $(z=0)$

$$\left[\frac{\partial a_1}{\partial x}-\beta\frac{\partial a_2}{\partial x}\right]=0, \quad \left[c_{\perp}^2\left(\frac{\partial a_1}{\partial z}+\frac{\partial a_2}{\partial x}\right)+\alpha\frac{\partial a_2}{\partial x}\right]=0 \tag{1.6}$$

$$\left[-m^{-1}p'+2c_{\perp}^2\frac{\partial a_2}{\partial z}\right]=0$$

$$\alpha\frac{\partial a_1^-}{\partial x}-\beta c_{\perp+}^2\left(\frac{\partial a_1^+}{\partial z}+\frac{\partial a_2^+}{\partial x}\right)-2c_{\perp+}^2\frac{\partial a_2^+}{\partial z}+m^{-1}p_+'=0$$

$$\alpha\equiv\frac{[2c_{\perp}^2(\varkappa_{11}-\varkappa_{22})]}{[\varkappa_{22}+\Delta_{22}]}, \quad \beta\equiv\frac{[\varkappa_{12}+\Delta_{12}]}{[\varkappa_{22}+\Delta_{22}]} \tag{1.7}$$

All the information about the deformed state of the system is contained in the two parameters α and β. In an equilibrium heterogeneous configuration with plane coherent boundary between the homogeneously deformed phases, the state of the phase, when the transformation ND is known, is uniquely determined by the state of the other phase from the formulas (see (9), (10) of /2/)

$$\varkappa_{11}^-=\varkappa_{11}^+-\Delta_{11}, \quad \varkappa_{22}^-=\varkappa_{22}^++\Delta_{11} \tag{1.8}$$

$$\varkappa_{12}^-=\varkappa_{12}^++\Delta_{21}+2(\chi-1)\varkappa_{(12)}^+, \quad \varkappa_{21}^-=\varkappa_{21}^+-\Delta_{21}$$

$$\sigma_{11}^-=(1-\chi^{-1})\sigma_{22}^++\chi^{-1}\sigma_{11}^+-4\mu_-\Delta_{11}, \quad \sigma_{12}^-=\sigma_{12}^+, \quad \sigma_{22}^-=\sigma_{22}^+$$

$$p^-=p^+-2[\mu]\varkappa_{22}^++2\mu_-\Delta_{11}=\frac{1}{2}\chi^{-1}(\sigma_{22}^+-\sigma_{11}^+)+\sigma_{22}^++2\mu_-\Delta_{11}$$

$$\sigma_{ij}\equiv-p\delta_{ij}+2\mu\varkappa_{(ij)}, \quad \chi\equiv\mu_+/\mu_-$$

Substituting (1.8) into (1.7), we obtain relations for the parameters α and β in terms of the given $\varkappa_{ij}^+$ or σ_{ij}^+:

$$\alpha=-\Delta^{-1}(2[c_{\perp}^2](\varkappa_{11}^+-\varkappa_{22}^+)+4c_{\perp-}^2\Delta_{11})= \tag{1.9}$$

$$-m^{-1}\Delta^{-1}((\sigma_{11}^+-\sigma_{22}^+)(1-\chi^{-1})+4\mu_-\Delta_{11}), \quad \Delta\equiv\Delta_{11}+\Delta_{22}$$

$$\beta=2\Delta^{-1}(\Delta_{(12)}+(\chi-1)\varkappa_{(12)}^+)= \tag{1.10}$$

$$\Delta^{-1}(2\Delta_{(12)}+(\chi-1)\sigma_{12}^+/\mu_+)$$

System (1.5), (1.6), like (1.3), has to be augmented by suitable conditions on the external boundary.

2. *Local stability of the coherent boundary.* We consider an actual two-phase configuration in which the domain $z>0$ is filled with the plus phase, and $z<0$ with the minus phase. Following the definition of /4/, the coherent boundary is stable if, to the solutions $a^i(x,z)$, $p'(x,z)$ of the spectral Problem (1.5), (1.6), which are exponentially damped in the relevant half-spaces and are oscillatory in the direction of the interphase boundary, there correspond non-negative eigenvalues π. The exponentially damped solutions of (1.5) in the upper and lower half-spaces are

$$a_1^{\pm}=\pm i(B_1^{\pm}\exp(\mp kz)+B_2^{\pm}\xi^{\pm}\exp(\mp k\xi^{\pm}z))\exp(-ikx) \tag{2.1}$$

$$a_2^{\pm}=(B_1^{\pm}\exp(\mp kz)+B_2^{\pm}\exp(\mp k\xi^{\pm}z))\exp(-ikx)$$

$$p_{\pm}'=-\pi mk^{-1}B_1^{\pm}\exp(k(\mp z-ix)), \quad \xi^{\pm}\equiv\sqrt{1-\pi k^{-2}c_{\perp\pm}^{-2}}$$

(k is the real wave number).

Substitution of (2.1) into condition (1.6) on the interphase boundary leads to a linear homogeneous system of algebraic equations in $B_1^{\pm}$, $B_2^{\pm}$. The condition for a non-trivial solution of Problem (1.5), (1.6) to exist leads to vanishing of the determinant of the system, which implies the following equation for the spectral value $q\ (q\equiv\pi k^{-2})$:

$$\xi_-R_+^2+\xi_+R_-^2-(Q_-^2+Q_+^2)\xi_+\xi_--\beta^2(4c_{\perp+}^4\xi_+-S_+^2+4c_{\perp-}^4\xi_--S_-^2)=0 \tag{2.2}$$

$$Q\equiv2c_{\perp}^2+\alpha, \quad R\equiv2c_{\perp}^2-q+\alpha, \quad S\equiv2c_{\perp}^2-q$$

Putting $q = 0$ and applying l'Hôpital's rule to the resulting indeterminate form, we arrive at the equation of the surface of neutral stability

$$^{1}/_{4}c_{\perp+}^{-2}c_{\perp-}^{-2}\alpha^2 - \beta^2 = 1 \tag{2.3}$$

This equation defines a hyperbola in the plane of the variables α, β. If the point (α_0, β_0) is outside a branch of the hyperbola (we have in mind the domains that contain the points $\beta = 0, \alpha \to \pm\infty$), then Eq.(2.2) must have a negative root q, i.e., the equilibrium configuration is unstable.

The complicated form of Eq.(2.2) prevents us making a full analysis of the signs of the roots for arbitrary α and β. Without dwelling on the proofs, we can indicate some properties of Eq.(2.2): for any value of $\alpha = \alpha_0$ $(\beta = \beta_0)$ we can find a value $\beta = \beta_0$ $(\alpha = \alpha_0)$ such that, for all β (all α) which satisfy the condition $|\beta| > \beta_0$ $(|\alpha| > \alpha_0)$, Eq.(2.2) has no negative roots (has a negative root); the coherent boundary then satisfies the necessary conditions for stability (the sufficient conditions for instability).

Using (1.9) and (1.10), we can rewrite the equation of the surface of neutral stability in the parameters $\varkappa_{ij}^+$ or σ_{ij}^+:

$$((\varkappa_{11}^+ - \varkappa_{22}^+)(\chi - 1) + 2\Delta_{11})^2 - 4\chi(\Delta_{(12)} + (\chi - 1)\varkappa_{(12)}^+)^2 = \chi\Delta^2 \tag{2.4}$$

$$\frac{1}{4}\mu_+^{-2}\chi((\sigma_{11}^+ - \sigma_{22}^+)(1 - \chi^{-1}) + 4\mu_+\chi^{-1}\Delta_{11})^2 - 4(\Delta_{(12)} + (\chi - 1)\sigma_{12}^+\mu_+^{-1})^2 = \Delta^2 \tag{2.5}$$

If $\chi \neq 1$ and $\Delta_{11} \neq -\Delta_{22}$, Eq.(2.5) defines in the space of parameters $(\sigma_{11}^+, \sigma_{22}^+, \sigma_{12}^+)$ a hyperbolic cylinder (if $\Delta_{11} = -\Delta_{22}$, the cylinder degenerates into two intersecting planes). Domains outside the cylinder correspond to unstable equilibria. If $\chi = 1$, the loss of stability depends on the NC (see (2.4) or (2.5)).

If we fix Δ_{ij}, σ_{ij}^+, and μ_+, while $\sigma_{12}^+ \neq 0$, then, as $\chi \to \infty$ $(\chi \to 0)$ the left-hand side of (2.5) is less than the right-hand side (greater than the right-hand side if $\sigma_{11}^+ - \sigma_{22}^+ \neq 4\mu_+\Delta_{11}$, which is equivalent to the condition $\sigma_{11}^- = \sigma_{22}^-$), and the corresponding equilibria will satisfy the necessary conditions for stability (the sufficient conditions for instability).

Let $\sigma_{12}^+ = 0$; we then see from (2.5) that, for sufficiently large χ, any non-hydrostatic stresses in the plus phase lead to instability of the two-phase equilibrium. This result agrees with that obtained in /6/ on the instability of the interphase boundary separating a non-hydrostatically stressed solid phase and its melt in the case of a slipping transition.

Consider the phase transformations that accompany an ND of cubic expansion-compression $(\Delta_{ij} = \delta\delta_{ij})$, with $\sigma_{12}^+ = 0$. If $\chi = 1$, then $\alpha = -2c_\perp^2$, and (2.2) has the roots $q = 0$ and $q_+ = c_\perp^2$, i.e., the system is stable. If $\chi < 1$ and $\sigma_{11}^+ = \sigma_{22}^+$ or $\chi > 1$ and $\sigma_{11}^- = \sigma_{22}^-$, then the interphase boundary is unstable (see (2.4), (2.5)). This is a typical feature of coherent boundaries: with an ND of cubic expansion-compression, a plane coherent boundary is always unstable, if, in the hydrostatic state, there is a phase with a lower shift modulus (this is also true for compressible phases). Notice that both phases cannot be simultaneously in the hydrostatically stressed state (see (1.8)).

The above sufficient conditions for instability can be used to analyse the local stability of heterogeneous systems with curved boundaries and inhomogeneously deformed phases, since, in the case of sufficiently short disturbances, the local curvature and inhomogeneity of the equilibrium configuration can be neglected. For instance, we know that, in the case of coherent transformations with ND of cubic expansion-compression, the equilibrium elliptic embryos of the new phase in elastic matrices are in the hydrostatically stressed state /1/. On the basis of our above necessary conditions for the stability of coherent boundaries, we can assert that the equilibrium configuration with an embryo is unstable, if the new phase has a lower shift modulus than the basic phase.

3. ***The stability of a two-phase periodic structure.*** Here, as the actual equilibrium configuration, we consider a heterogeneous system consisting of alternating layers of plus and minus phase. The layer boundaries in the initial one-phase configuration are assumed to be parallel, and the thicknesses $2H_+$ and $2H$ of the layers (pre-images) are the same for each phase.

Using (1.8), and given the equilibrium homogeneous stressed state in the layers of plus phase, we can calculate the state in the layers of minus phase. The stressed states of all layers of a given phase are the same, so that the values of the parameters α and β are the same for all interphase boundaries.

Consider the stability of a periodic structure with respect to variations (disturbances) of the field of auxiliary displacements $a_i(x, z)$ and $p'(x, z)$ of a periodic kind (the x axis is along a boundary), i.e.,

$$a_i(x, z) = a_i(x, z \pm 2(H_+ + H_-)) \tag{3.1}$$

$$p'(x, z) = p'(x, z \pm 2(H_+ + H_-))$$

We will say that a heterogeneous two-phase periodic system with coherent interphase boundaries is stable if, corresponding to the solution a_i, p' of the spectral Problem (1.5), (1.6), which are periodic along the z axis and are oscillatory along the interphase boundaries, we have non-negative eigenvalues π.

When studying the equilibrium and stability of such systems, we can distinguish in a natural way a "periodic cell" of thickness $2(H_+ + H_-)$, consisting of two adjacent layers. All the characteristics of the periodic system are repeated on moving from one cell to another. Consider the periodic cell $z \in [-2H_-, 2H_+]$ (for clarity, let $z \in [0, 2H_+]$ be a plus phase layer). The general solution of the system of differential Eqs.(1.5) in the cell can be written as the sum of symmetric and antisymmetric **disturbances**. The symmetric disturbances are

$$\begin{aligned} f_1^{\pm} &= i(F_1^{\pm} \operatorname{ch} k(z \mp H_{\pm}) + F_2^{\pm}\xi_{\pm} \operatorname{ch} k\xi_{\pm}(z \mp H_{\pm}))\, e^{-ikx} \\ f_2^{\pm} &= -(F_1^{\pm} \operatorname{sh} k(z \mp H_{\pm}) + F_2^{\pm} \operatorname{sh} k\xi_{\pm}(z \mp H_{\pm}))\, e^{-ikx} \\ p'_{f\pm} &= -k^{-1}\pi m F_1^{\pm} \operatorname{ch} k(z \mp H)\, e^{-ikx} \end{aligned} \qquad (3.2)$$

For the antisymmetric **disturbances** $g_{1,2}^{\pm}$ and $p'_{g\pm}$ the cosines and sines in each of (3.2) are interchanged, and instead of the constants $F_{1,2}^{\pm}$ we take the constants $G_{1,2}^{\pm}$.

The functions f_i^- and p_{f-}' in (3.2) are taken for $z \in [-2H_-, 0]$, and the functions f_i^+ and p_{f+}' for $z \in [0, 2H_+]$. In the layer $z \in [0, 2H_+]$, the vector $\mathbf{f}_+ = \{f_1^+, f_2^+\}$ defines **disturbances** of the displacement field which are symmetrical about the axis $z = H_+$, and the vector $\mathbf{g}_+$ defines the antisymmetric disturbances. In particular, on the layer boundaries $z = 0$ and $z = 2H_+$, we have

$$\begin{aligned} f_1^+(x, 0) &= f_1^+(x, 2H_+), \quad f_2^+(x, 0) = -f_2^+(x, 2H_+) \\ p'_{f+}(x, 0) &= p'_{f+}(x, 2H_+), \quad p'_{g+}(x, 0) = -p'_{g+}(x, 2H_+) \\ g_1^+(x, 0) &= -g_1^+(x, 2H_+), \quad g_2^+(x, 0) = g_2^+(x, 2H_+) \end{aligned} \qquad (3.3)$$

The properties of $f_{1,2}^-$, $g_{1,2}^-$, p_{f-}' and p_{g-}' are similar.

We substitute (3.1) and (3.2) into the system of boundary conditions (1.6) with $z = 0$. We obtain a first group of algebraic equations in $F_{1,2}^{\pm}$ and $G_{1,2}^{\pm}$. A second group is obtained by considering the boundary relations (1.6) with $z = 2H_+$. As a_i^+ and p_+' we take $a_i^+(x, 2H_+)$, $p_+'(x, 2H_+)$ of (3.2). Using the periodicity condition (3.1), we take as a_i^- an p_-' of the next cell their values $a_i^-(x, -2H_-)$, $p_-'(x, -2H_-)$. Together, the two groups of linear homogeneous algebraic equations form a system of eight equations for the eight constants $F_{1,2}^{\pm}$ and $G_{1,2}^{\pm}$.

We confine ourselves to the deformed states of the periodic structure in which the parameter β vanishes (for this, it suffices to put e.g., $\Delta_{(12)} = 0$ and $\sigma_{12}^+ = 0$). Then, after equivalent transformations that take account of properties (3.3), our system splits into two independent subsystems, the first for $F_{1,2}^{\pm}$, and the second for $G_{1,2}^{\pm}$.

The condition for a non-trivial solutions $f_i^{\pm}$, $p'_{f\pm}$ of the spectral Problem (1.5), (1.6) to exist is the vanishing of the determinant D_f of the first subsystem, and for the solution $g_i^{\pm}$, $p'_{g\pm}$, the vanishing of D_g of the second subsystem. On evaluating D_f, we arrive at an equation for the spectral parameter q $(h = kH)$

$$D_f = q(\xi_+\xi_-(Q_-^2 \operatorname{th} h_- + Q_+^2 \operatorname{th} h_+) - \xi_+ R_-^2 \operatorname{th} h_-\xi_- - \xi_- R_+^2 \operatorname{th} h_+\xi_+) = 0 \qquad (3.4)$$

If we deal with the indeterminate form with $q = 0$ and equate the resulting expression to zero, we obtain the equation of the surface of neutral stability for symmetric disturbances

$$\begin{gathered} (q^{-2}D_f)_{q=0} = -{}^1/_2\alpha^2(c_{\perp-}^{-2}(\operatorname{th} h_- - t_-) + c_{\perp+}^{-2}(\operatorname{th} h_+ - t_+)) + 2\alpha(t_- + t_+) + \\ 2(c_{\perp-}^2(\operatorname{th} h_- + t_-) + c_{\perp+}^2(\operatorname{th} h_+ + t_+)) = 0 \\ t_{\pm} \equiv h_{\pm}(1 - \operatorname{th}^2 h_{\pm}) \quad ((D_f)_{q=0} = (q^{-1}D_f)_{q=0} = 0) \end{gathered}$$

We can similarly show that the cotangents instead of the tangents appear in the dispersion Eq.(3.4) and the equation of the surface of neutral stability (3.5) for antisymmetric disturbances.

Note that Eq.(3.5) always has two real roots of different signs $\alpha_{1,2}^f$ $(\alpha_1^f < 0 < \alpha_2^f)$. For the case $h_+ = h_- = h$, curves of the dimensionless quantities $\bar{\alpha}_{1,2}^f$ and $\bar{\alpha}_{1,2}^g$ against h for $\chi = 1$ and $\chi = 10$ are shown in Fig.1 (here we put $\bar{\alpha} = {}^1/_2\alpha c_{\perp+}^{-1} c_{\perp-}^{-1}$). The broken curves refer to $\bar{\alpha}^{f}_{1,2}$ and the continuous ones to $\bar{\alpha}^{g}_{1,2}$. If $\chi = 1$, then $\bar{\alpha}_1{}^f = \bar{\alpha}_1{}^g = -1$ for any h.

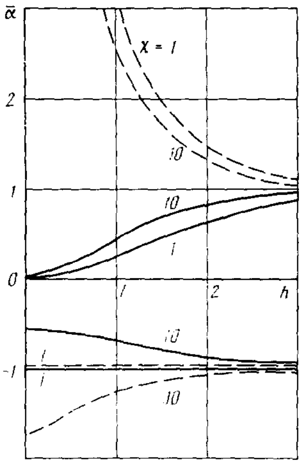

Fig.1

If $\alpha < \alpha_1{}^f$ or $\alpha_2{}^f < \alpha$, Eq.(3.4) always has a negative root q. A similar assertion can be made for antisymmetric disturbances.

To sum up, if the stress-deformed state in the layers of plus phase is such that $\beta = 0$, while $\alpha \in]-\infty, \alpha_1{}^f[\bigcup]\alpha_2{}^f, +\infty[$ ($\alpha \in]-\infty, \alpha_1{}^g[\bigcup]\alpha_2{}^g, +\infty[$), then the equilibrium in the two-phase periodic structure will be unstable with respect to symmetric (antisymmetric) disturbances.

Consider some asymptotic cases. When $h_+ \sim h_- \to +\infty$, we arrive at the results of Sect.2 regardless of the type of disturbances.

Let $h_+ \sim h_- \to 0$ (this refers physically to the case of long disturbances k^{-1} and finite layer thicknesses $H_\pm$, or of finite disturbance lengths and small thicknesses). The system will be unstable with respect to symmetric disturbances $\bar{\alpha} \in]-\infty, -{}^1/_2(\sqrt{\chi} + 1/\sqrt{\chi})[$ (the root $\bar{\alpha}_2{}^f$ becomes infinite), or with respect to antisymmetric disturbances if $\bar{\alpha} \in]-\infty, -2(\sqrt{\chi} + 1/\sqrt{\chi})^{-1}[\bigcup]0, +\infty[$.

It can be seen from Fig.1 that long-wave disturbances can lead to loss of stability of heterogeneous coherent equilibria, in spite of the fact that the necessary conditions for the stability of the individual interphase boundaries are satisfied (these conditions are here $|\bar{\alpha}| < 1$).

If $h_+ \sim 1, h_- \to 0$ (this refers to a two-phase structure with thin periodic coherent embryos), the domain of instability for antisymmetric disturbances is the union of intervals $]-\infty, -1/\sqrt{\chi}[\bigcup]0, +\infty[$, or for symmetric disturbances, the union $-]-\infty, -\sqrt{\chi}[\bigcup]\sqrt{\chi}\,(\mathrm{sh}\,2h_+ + 2h_+)(\mathrm{sh}\,2h_+ - 2h_+)^{-1}, +\infty[$.

4. *The problem of the stability of a secluded plane inclusion of new phase in an unbounded elastic matrix.* In the initial configuration, let the layer $z \in]-2H_-, 0[$ be the pre-image of the embryo of a new minus phase, and let the domains $z \in]-\infty, -2H_-[\bigcup]0, +\infty[$ be the pre-images of the half-spaces filled with basic plus phase. It follows from (1.8) that the stress-deformed states in the half-spaces of plus phase are the same and uniquely define the states in the embryo. The parameters α and β are the same for both interphase boundaries.

We will say that the equilibrium of the coherent embryo of new phase in an unbounded elastic matrix is stable if, corresponding to the solutions a_i, p' of the spectral Problem (1.5), (1.6), which are exponentially damped within the half-spaces of new phases, and are oscillatory along the embryo boundaries, we have non-negative eigenvalues q. As in the problem of the stability of a periodic structure, with $\beta = 0$ system (1.5), (1.6) has solutions of symmetric and antisymmetric types. Arguments similar to those in Sects.2 and 3 lead to a dispersion equation for the parameter q and an equation of the surface of neutral stablity. For the solutions of symmetric type, these equations are the same as (3.4) and (3.5), if h_+ tends to infinity in the latter. For the antisymmetric solutions, instead of tangents in these equations we have cotangents.

The equilibrium of the new phase embryo in the elastic matrix becomes unstable if $\alpha < \alpha_1$ or $\alpha_2 < \alpha$.

In Fig.2 we show curves of $\bar{\alpha}^{f}_{1,2}$ and $\bar{\alpha}^{g}_{1,2}$ against h for $\chi = 0, 1, 1, 10$ (continuous curve for $\bar{\alpha}^{g}_{1,2}$ broken for $\bar{\alpha}^{f}_{1,2}$). If $\chi = 1$, then $\bar{\alpha}_1{}^f = \bar{\alpha}_1{}^g = -1$. If $h_- \to 0$, then $\bar{\alpha}_1{}^f = -\sqrt{\chi}, \bar{\alpha}_2{}^f = \sqrt{\chi}, \bar{\alpha}_1{}^g = -1/\sqrt{\chi}, \bar{\alpha}_2{}^g = 0$. We can see from Fig.2 that stability is lost if the embryo thickness increases for $|\bar{\alpha}| > 1$, or if it decreases for $0 < \bar{\alpha}$ or $\bar{\alpha} < \max(-\sqrt{\chi}, -1/\sqrt{\chi})$.

5. *The stability of a coherent embryo of new phase on the free surface of an elastic half-space.* In the initial configuration, let the domain $z \in]0, +\infty[$ correspond to the pre-image of the half-space filled by basic plus phase, and let $z \in]-H, 0[$ correspond to the pre-image of the layer of new minus phase. We will assume that, in the actual equlibrium configuration, the principal directions of the tensor $\Delta_{(ij)}$ are the same as the x and z axes. On tne outer boundary of the emoryo $(z = -H)$, constant zero pressure is maintained. System (1.8) is augmented by the relations

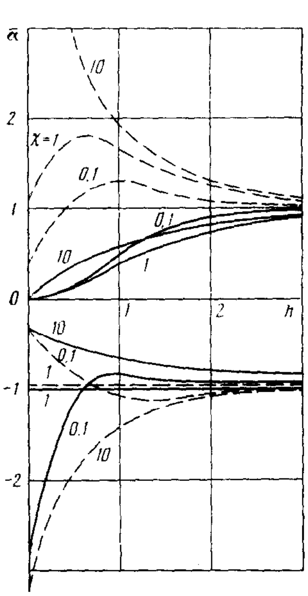

Fig.2

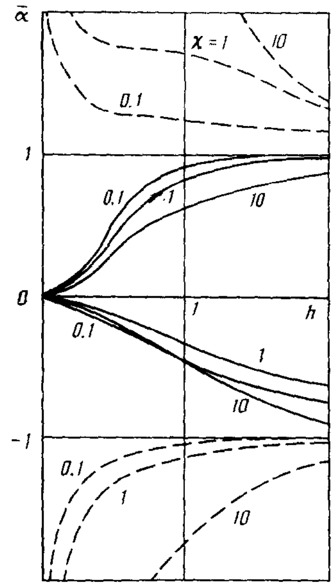

Fig.3

$$\sigma_{22}^- = 0, \quad \sigma_{12}^- = 0 \tag{5.1}$$

From (1.8), (5.1) and (1.9), we obtain relations for the homogeneous equilibrium stressed state in the embryo and for the parameters α and β, given σ_{11}^+, Δ_{11} and Δ_{22} (it follows from (1.8) and (5.1) that $\sigma_{22}^+ = 0$ and $\sigma_{12}^+ = 0$)

$$\sigma_{11}^- = \chi^{-1}\sigma_{11}^+ - 4\mu_-\Delta_{11}, \quad \sigma_{12}^- = 0, \quad \sigma_{22}^- = 0$$
$$\alpha = -m^{-1}\Delta^{-1}(\sigma_{11}^+(1-\chi^{-1}) + 4\mu_-\Delta_{11}), \quad \beta = 0 \tag{5.2}$$

When studying the stability of this equilibrium, the spectral problem (1.5), (1.6) is augmented by the following equations, obtained from the condition that the pressure on the outer boundary of the embryo be constant ($z = -H$):

$$\frac{\partial a_1^-}{\partial z} + \frac{\partial a_2^-}{\partial x} = 0, \quad p_-' + 2\mu_- \frac{\partial a_2^-}{\partial z} = 0 \tag{5.3}$$

The equilibrium embryo is said to be stable if, corresponding to the solutions of Problem (1.5), (1.6), (5.3), which are exponentially damped as $z \to +\infty$, and are oscillatory along the interphase boundary, we have non-negative values of the parameter q. An analysis similar to that in Sects.2 and 3, leads to an equation for q, and a relation for the critical deformations:

$$\xi_-(Q_+^2\xi_+ - R_+^2)(4c_{\perp -}^4\xi_- \operatorname{ch} h \operatorname{sh} h\xi_- + S_- \operatorname{sh} h \operatorname{ch} h\xi_-) + 4\xi_+\xi_- c_{\perp -}^2 Q_- R_- S_- -$$
$$\xi_+\xi_- \operatorname{ch} h \operatorname{ch} h\xi_- (4c_{\perp -}^4 R_-^2 - Q_-^2 S_-^2) + \xi_+ \operatorname{sh} h \operatorname{sh} h\xi_- (4c_{\perp -}^4 \xi_-^2 Q_-^2 + R_-^2 S_-^2) = 0 \tag{5.4}$$

$$-\alpha^2(\chi^{-1}(\operatorname{sh} h \operatorname{ch} h - h) + \operatorname{ch}^2 h + h^2) + 4\alpha c_{\perp -}^2 h^2 + 4c_{\perp -}^4(\chi(\operatorname{sh} h \operatorname{ch} h - h) +$$
$$\operatorname{sh}^2 h - h^2) = 0 \tag{5.5}$$

For fixed χ and h, Eq.(5.5) always has two real roots of opposite signs $\alpha_{1,2}$ $(\alpha_1 < 0 < \alpha_2)$. For $\alpha \in]-\infty, \alpha_1[\cup]\alpha_2, +\infty[$, Eq.(5.4) has a negative root q. The present equilibrium two-phase configuration is therefore unstable if, given the stresses σ_{11}^+ expanding (or compressing) along the interphase boundary, and given the transformation deformation Δ_{11}, Δ_{22}, the embryo thickness H, and the disturbance length k^{-1}, the parameter α obtained from (5.2) lies in the domain $]-\infty, \alpha_1[\cup]\alpha_2, +\infty[$.

The continuous curves in Fig.3 represent $\bar{\alpha}_{1,2}$ against h. As $h \to +\infty$ (5.4) splits into two equations. One is the Rayleigh equation for the surface waves in the isotropic half-space of minus phase. We know /5/ that there are only real positive roots corresponding to this equation. The second is Eq.(2.2) with $\beta = 0$. Accordingly, $\bar{\alpha}_{1,2} \to \mp 1$. As $h \to 0$, we can write $\bar{\alpha}_{1,2}$ asymptotically in terms of h:

$$\bar{\alpha}_{1,2} = \mp {}^2/_3 \chi h^{3/2} + o(h^{3/2})$$

i.e., the threshold values of the parameter α at which instability occurs tend to zero (**Fig.3**). Consequently, in the case of a non-zero jump of the principal stresses in the tangential direction to the interphase boundary ($[\sigma_{11}] = -m\Delta\alpha$), a reduction of the embryo thickness leads to a loss of stability.

6. The stability of a coherent embryo on the boundary of the elastic half-space of the basic phase and rigid wall. Let $\Delta_{(12)} = 0$, $\sigma_{12}{}^+ = 0$ (the parameter β then vanishes). On the outer embryo boundary, we take the condition for the total displacements of the material particles to vanish. Using expansions (1.1), this leads to the relations

$$v_1{}^- + \Delta_{11}x = 0, \quad v_2{}^- - \Delta_{22}H = 0 \tag{6.1}$$

The presence of a rigid wall in the configuration constrains the choice of $\sigma_{11}{}^+$ and $\sigma_{22}{}^+$. It follows from (1.8) and (6.1) that, in an actual configuration, only the hydrostatic state of the plus phase is possible: $\sigma_{11}{}^+ = \sigma_{22}{}^+ = -p_+$. Given $p_+, \Delta_{11}, \Delta_{22}$, the homogeneous stressed state in the embryo and the parameter α are given by (1.8), (1.9), (6.1), in accordance with

$$\begin{gathered}\sigma_{11}{}^- = -p_+ - 4\mu_-\Delta_{11}, \quad \sigma_{22}{}^- = -p_+ \\ \sigma_{12}{}^- = 0, \quad \alpha = 4c_{\perp -}^2\Delta_{11}\Delta^{-1}\end{gathered} \tag{6.2}$$

The same definition of stability is taken as in Sect.5. Instead of (5.3), with $z = -H$ we have the condition

$$a_1{}^- = a_2{}^- = 0 \tag{6.3}$$

The corresponding equation for the eigenvalue q of specral Problem (1.5), (1.6), (6.3), and the equation of the surface of neutral stability, are

$$\begin{gathered}-q^2\xi_+\xi_- - \xi_+R_-{}^2(\xi_- \operatorname{ch} h \operatorname{ch} h\xi_- - \operatorname{sh} h \operatorname{sh} h\xi_- - \xi_-) + \\ \xi_-(Q_+{}^2\xi_+ - R_+{}^2)(\xi_- \operatorname{ch} h \operatorname{sh} h\xi_- - \operatorname{sh} h \operatorname{ch} h\xi_-) - \\ \xi_+\xi_-Q_-{}^2(\operatorname{ch} h \operatorname{ch} h\xi_- - \xi_- \operatorname{sh} h \operatorname{sh} h\xi_- - 1) = 0\end{gathered} \tag{6.4}$$

$$\begin{gathered}\alpha^2(h + \operatorname{sh} h \operatorname{ch} h + \chi(\operatorname{sh}^2 h - h^2)) - 4\alpha h^2 c_{\perp +}^2 - c_{\perp -}^2(h + \operatorname{sh} h \operatorname{ch} h + \chi^{-1}(\operatorname{ch}^2 h + \\ h^2)) = 0\end{gathered} \tag{6.5}$$

Eq.(6.5) has two real roots of different signs α_1 and α_2 ($\alpha_1 < 0 < \alpha_2$).

The broken curves in Fig.3 show $\bar{\alpha}_{1,2}$ against h ($\bar{\alpha}_1 \to -1$, $\bar{\alpha}_2 \to 1$ as $h \to +\infty$; $\bar{\alpha}_1 \to -\infty$, $\bar{\alpha}_2 \to +\infty$ as $h \to 0$). If $\alpha < \alpha_1$ or $\alpha_2 < \alpha$, Eq.(6.4) has a negative root q, and the equilibrium configuration is then unstable. Note that, as distinct from Sect.5, the stability does not depend on p_+.

We see from Fig.3 that, if the necessary conditions for local stability of Sect.2 are not satisfied, then, no matter what the fixed length of the disturbance, the system always loses stability when the thickness of the coherent embryo increases.

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